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The β-function in duality-covariant noncommutative ϕ^4-theory

Harald Grosse[1] and Raimar Wulkenhaar[2]

[1] *Institut für Theoretische Physik, Universität Wien*
Boltzmanngasse 5, A-1090 Wien, Austria

[2] *Max-Planck-Institut für Mathematik in den Naturwissenschaften*
Inselstraße 22-26, D-04103 Leipzig, Germany

Abstract

We compute the one-loop β-functions describing the renormalisation of the coupling constant λ and the frequency parameter Ω for the real four-dimensional duality-covariant noncommutative ϕ^4-model, which is renormalisable to all orders. The contribution from the one-loop four-point function is reduced by the one-loop wavefunction renormalisation, but the β_λ-function remains non-negative. Both β_λ and β_Ω vanish at the one-loop level for the duality-invariant model characterised by $\Omega = 1$. Moreover, β_Ω also vanishes in the limit $\Omega \to 0$, which defines the standard noncommutative ϕ^4-quantum field theory. Thus, the limit $\Omega \to 0$ exists at least at the one-loop level.

[1] harald.grosse@univie.ac.at
[2] raimar.wulkenhaar@mis.mpg.de

1 Introduction

For many years, the renormalisation of quantum field theories on noncommutative $\mathbb{R}^4$ has been an open problem [1]. Recently, we have proven in [2] that the real duality-covariant ϕ^4-model on noncommutative $\mathbb{R}^4$ is renormalisable to all orders. The duality transformation exchanges positions and momenta [3],

$$\hat{\phi}(p) \leftrightarrow \pi^2\sqrt{|\det\theta|}\;\phi(x)\;, \qquad\qquad p_\mu \leftrightarrow \tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}x^\nu\;, \tag{1}$$

where $\hat{\phi}(p_a) = \int d^4x\; \mathrm{e}^{(-1)^a \mathrm{i} p_{a,\mu}x_a^\mu}\phi(x_a)$. The subscript a refers to the cyclic order in the $\star$-product. The duality-covariant noncommutative ϕ^4-action is given by

$$\begin{aligned} S[\phi;\mu_0,\lambda,\Omega] := \int d^4x\; \Big(\frac{1}{2}(\partial_\mu\phi)\star(\partial^\mu\phi) + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu_0^2}{2}\phi\star\phi \\ + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x)\;. \end{aligned} \tag{2}$$

Under the transformation (1) one has

$$S\big[\phi;\mu_0,\lambda,\Omega\big] \mapsto \Omega^2 S\Big[\phi;\frac{\mu_0}{\Omega},\frac{\lambda}{\Omega^2},\frac{1}{\Omega}\Big]\;. \tag{3}$$

In the special case $\Omega=1$ the action $S[\phi;\mu_0,\lambda,1]$ is invariant under the duality (1). Moreover, $S[\phi;\mu_0,\lambda,1]$ can be written as a standard matrix model which is closely related to an exactly solvable model [4].

Knowing that the action (2) gives rise to a renormalisable quantum field theory [2], it is interesting to compute the β_λ and β_Ω functions which describe the renormalisation of the coupling constant λ and of the oscillator frequency Ω. Whereas we have proven the renormalisability in the Wilson-Polchinski approach [5, 6] adapted to non-local matrix models [7], we compute the one-loop β_λ and β_Ω functions by standard Feynman graph calculations. Of course, these are Feynman graphs parametrised by matrix indices instead of momenta. We rely heavily on the power-counting behaviour proven in [2], which allows us to ignore in the β-functions all non-planar graphs and the detailed index dependence of the planar two- and four-point graphs. Thus, only the lowest-order (discrete) Taylor expansion of the planar two- and four-point graphs can contribute to the β-functions. This means that we cannot refer to the usual symmetry factors of commutative ϕ^4-theory so that we have to carefully recompute the graphs.

We obtain interesting consequences for the limiting cases $\Omega=1$ and $\Omega=0$ as discussed in Section 5.

2 Definition of the model

The noncommutative $\mathbb{R}^4$ is defined as the algebra $\mathbb{R}^4_\theta$ which as a vector space is given by the space $\mathcal{S}(\mathbb{R}^4)$ of (complex-valued) Schwartz class functions of rapid decay, equipped

with the multiplication rule

$$(a\star b)(x) = \int \frac{d^4k}{(2\pi)^4} \int d^4y\; a(x{+}\tfrac{1}{2}\theta{\cdot}k)\, b(x{+}y)\, \mathrm{e}^{\mathrm{i}k\cdot y}\;, \qquad (4)$$
$$(\theta{\cdot}k)^\mu = \theta^{\mu\nu}k_\nu\;,\quad k{\cdot}y = k_\mu y^\mu\;,\quad \theta^{\mu\nu} = -\theta^{\nu\mu}\;.$$

We place ourselves into a coordinate system in which the only non-vanishing components $\theta_{\mu\nu}$ are $\theta_{12} = -\theta_{21} = \theta_{34} = -\theta_{42} = \theta$. We use an adapted base

$$b_{mn}(x) = f_{m^1n^1}(x^1,x^2)\, f_{m^2n^2}(x^3,x^4)\;, \qquad m = {m^1 \atop m^2} \in \mathbb{N}^2\,,\; n = {n^1 \atop n^2} \in \mathbb{N}^2\;, \qquad (5)$$

where the base $f_{m^1n^1}(x^1,x^2) \in \mathbb{R}^2_\theta$ is given in [8]. This base satisfies

$$(b_{mn}\star b_{kl})(x) = \delta_{nk}b_{ml}(x)\;, \qquad \int d^4x\, b_{mn}(x) = 4\pi^2\theta^2\delta_{mn}\;. \qquad (6)$$

According to [2], the duality-covariant ϕ^4-action (2) expands as follows in the matrix base (5):

$$S[\phi;\mu_0,\lambda,\Omega] = 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \Big(\frac{1}{2}G_{mn;kl}\phi_{mn}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)\;, \qquad (7)$$

where $\phi(x) = \sum_{m,n\in\mathbb{N}^2}\phi_{mn}b_{mn}(x)$ and

$$\begin{aligned} G_{mn;kl} = {} & \Big(\mu_0^2 + \frac{2}{\theta}(1{+}\Omega^2)(m^1{+}n^1{+}m^2{+}n^2{+}2)\Big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\ & - \frac{2}{\theta}(1{-}\Omega^2)\Big(\big(\sqrt{(n^1{+}1)(m^1{+}1)}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1} + \sqrt{n^1m^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} \\ & + \big(\sqrt{(n^2{+}1)(m^2{+}1)}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2} + \sqrt{n^2m^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1}\Big)\;. \end{aligned} \qquad (8)$$

The quantum field theory is defined by the partition function

$$Z[J] = \int \Big(\prod_{a,b\in\mathbb{N}^2} d\phi_{ab}\Big)\, \exp\big(-S[\phi] - 4\pi^2\theta^2 \sum_{m,n\in\mathbb{N}^2}\phi_{mn}J_{nm}\big)\;. \qquad (9)$$

For the free theory defined by $\lambda = 0$ in (7), the solution of (9) is given by

$$Z[J]\big|_{\lambda=0} = Z[0]\,\exp\Big(4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2}J_{mn}\Delta_{mn;kl}J_{kl}\Big)\;, \qquad (10)$$

where the propagator Δ is defined as the inverse of the kinetic matrix G:

$$\sum_{k,l\in\mathbb{N}^2} G_{mn;kl}\Delta_{lk;sr} = \sum_{\in\mathbb{N}^2}\Delta_{nm;lk}G_{kl;rs} = \delta_{mr}\delta_{ns}\,. \qquad (11)$$

We have derived the propagator in [2]:

$$
\begin{aligned}
&\Delta_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}} = \frac{\theta}{2(1{+}\Omega)^2}\delta_{m^1+k^1,n^1+l^1}\delta_{m^2+k^2,n^2+l^2} \\
&\times \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{\min(m^1+l^1,n^1+k^1)}{2}} \sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{\min(m^2+l^2,n^2+k^2)}{2}} B\big(1{+}\tfrac{\mu_0^2\theta}{8\Omega}{+}\tfrac{1}{2}(m^1{+}m^2{+}k^1{+}k^2){-}v^1{-}v^2, 1{+}2v^1{+}2v^2\big) \\
&\times {}_2F_1\left(\begin{matrix} 1{+}2v^1{+}2v^2\,,\; \frac{\mu_0^2\theta}{8\Omega}{-}\frac{1}{2}(m^1{+}m^2{+}k^1{+}k^2){+}v^1{+}v^2 \\ 2{+}\frac{\mu_0^2\theta}{8\Omega}{+}\frac{1}{2}(m^1{+}m^2{+}k^1{+}k^2){+}v^1{+}v^2 \end{matrix}\middle| \frac{(1{-}\Omega)^2}{(1{+}\Omega)^2}\right) \\
&\times \prod_{i=1}^{2}\sqrt{\binom{n^i}{v^i{+}\frac{n^i-k^i}{2}}\binom{k^i}{v^i{+}\frac{k^i-n^i}{2}}\binom{m^i}{v^i{+}\frac{m^i-l^i}{2}}\binom{l^i}{v^i{+}\frac{l^i-m^i}{2}}}\left(\frac{(1{-}\Omega)^2}{(1{+}\Omega)^2}\right)^{v^i} . \qquad (12)
\end{aligned}
$$

Here, $B(a,b)$ is the Beta-function and ${}_2F_1\big(\begin{smallmatrix} a,b \\ c\end{smallmatrix}\big|z\big)$ the hypergeometric function.

As usual we solve the interacting theory perturbatively:

$$
\begin{aligned}
Z[J] &= Z[0]\exp\Big(-V\Big[\frac{\partial}{\partial J}\Big]\Big)\exp\Big(4\pi^2\theta^2\sum_{m,n,k,l\in\mathbb{N}^2}\frac{1}{2}J_{mn}\Delta_{mn;kl}J_{kl}\Big) , \\
V\Big[\frac{\partial}{\partial J}\Big] &:= \frac{\lambda}{4!(4\pi^2\theta^2)^3}\sum_{m,n,k,l\in\mathbb{N}^2}\frac{\partial^4}{\partial J_{ml}\,\partial J_{lk}\,\partial J_{kn}\,\partial J_{nm}} . \qquad (13)
\end{aligned}
$$

It is convenient to pass to the generating functional of connected Green's functions, $W[J]=\ln Z[J]$:

$$
\begin{aligned}
W[J] &= \ln Z[0] + W_{\text{free}}[J] + \ln\Big(1+\mathrm{e}^{-W_{\text{free}}[J]}\Big(\exp\Big(-V\Big[\frac{\partial}{\partial J}\Big]\Big)-1\Big)\mathrm{e}^{W_{\text{free}}[J]}\Big) , \\
W_{\text{free}}[J] &:= 4\pi^2\theta^2\sum_{m,n,k,l\in\mathbb{N}^2}\frac{1}{2}J_{mn}\Delta_{mn;kl}J_{kl} . \qquad (14)
\end{aligned}
$$

In order to obtain the expansion in λ one has to expand $\ln(1{+}x)$ as a power series in x and $\exp(-V)$ as a power series in V. By Legendre transformation we pass to the generating functional of one-particle irreducible (1PI) Green's functions:

$$
\Gamma[\phi^{c\ell}] := 4\pi^2\theta^2\sum_{m,n\in\mathbb{N}^2}\phi^{c\ell}_{mn}J_{nm} - W[J] , \qquad (15)
$$

where J has to be replaced by the inverse solution of

$$
\phi^{c\ell}_{mn} := \frac{1}{4\pi^2\theta^2}\frac{\partial W[J]}{\partial J_{nm}} . \qquad (16)
$$

3 Renormalisation group equation

The computation of the expansion coefficients

$$\Gamma_{m_1n_1;\dots;m_Nn_N} := \frac{1}{N!}\frac{\partial^N\Gamma[\phi^{c\ell}]}{\partial\phi^{c\ell}_{m_1n_1}\dots\partial\phi^{c\ell}_{m_Nn_N}} \tag{17}$$

of the effective action involves possibly divergent sums over undetermined loop indices. Therefore, we have to introduce a cut-off $\mathcal{N}$ for all loop indices. According to [2], the expansion coefficients (17) can be decomposed into a relevant/marginal and an irrelevant piece. As a result of the renormalisation proof, the relevant/marginal parts have—after a rescaling of the field amplitude—the same form as the initial action (2), (7) and (8), now parametrised by the "physical" mass, coupling constant and oscillator frequency:

$$\Gamma_{\text{rel/marg}}[\mathcal{Z}\phi^{c\ell}] = S\big[\phi^{c\ell};\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}}\big] \; . \tag{18}$$

In the renormalisation process, the physical quantities μ^2_{phys}, λ_{phys} and Ω_{phys} are kept constant with respect to the cut-off $\mathcal{N}$. This is achieved by starting from a carefully adjusted initial action $S\big[\mathcal{Z}[\mathcal{N}]\phi,\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}]\big]$, which gives rise to the bare effective action $\Gamma[\phi^{c\ell};\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}],\mathcal{N}]$. Expressing the bare parameters μ_0,λ,Ω as a function of the physical quantities and the cut-off, the expansion coefficients of the renormalised effective action

$$\Gamma^R[\phi^{c\ell};\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}}] := \Gamma\big[\mathcal{Z}[\mathcal{N}]\phi^{c\ell},\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}],\mathcal{N}\big]\Big|_{\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}}=\text{const}} \tag{19}$$

are finite and convergent in the limit $\mathcal{N}\to\infty$. In other words,

$$\lim_{\mathcal{N}\to\infty}\mathcal{N}\frac{d}{d\mathcal{N}}\Big(\mathcal{Z}^N[\mathcal{N}]\Gamma_{m_1n_1;\dots;m_Nn_N}[\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}],\mathcal{N}]\Big) = 0 \; . \tag{20}$$

This implies the renormalisation group equation

$$\lim_{\mathcal{N}\to\infty}\Big(\mathcal{N}\frac{\partial}{\partial\mathcal{N}} + N\gamma + \mu_0^2\beta_{\mu_0}\frac{\partial}{\partial\mu_0^2} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_\Omega\frac{\partial}{\partial\Omega}\Big)\Gamma_{m_1n_1;\dots;m_Nn_N}[\mu_0,\lambda,\Omega,\mathcal{N}] = 0 \; , \tag{21}$$

where

$$\beta_{\mu_0} = \frac{1}{\mu_0^2}\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\mu_0^2[\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}},\mathcal{N}]\Big) \; , \tag{22}$$

$$\beta_\lambda = \mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\lambda[\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}},\mathcal{N}]\Big) \; , \tag{23}$$

$$\beta_\Omega = \mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\Omega[\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}},\mathcal{N}]\Big) \; , \tag{24}$$

$$\gamma = \mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\ln\mathcal{Z}[\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}},\mathcal{N}]\Big) \; . \tag{25}$$

4 One-loop computations

Defining $(\Delta J)_{mn} := \sum_{p,q\in\mathbb{N}^2} \Delta_{mn;pq} J_{pq}$ we write (parts of) the generating functional of connected Green's functions up to second order in λ:

$$
\begin{aligned}
W[J] = \ln Z[0] &+ 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2} J_{mn}\Delta_{mn;kl}J_{kl} \\
&- (4\pi^2\theta^2)\frac{\lambda}{4!} \sum_{m,n,k,l\in\mathbb{N}^2} \Big\{ (\Delta J)_{ml}(\Delta J)_{lk}(\Delta J)_{kn}(\Delta J)_{nm} \\
&\qquad + \frac{1}{4\pi^2\theta^2}\Big(\Delta_{nm;kn}(\Delta J)_{ml}(\Delta J)_{lk} + \Delta_{kn;lk}(\Delta J)_{nm}(\Delta J)_{ml} \\
&\qquad\qquad + \Delta_{nm;ml}(\Delta J)_{lk}(\Delta J)_{kn} + \Delta_{lk;ml}(\Delta J)_{kn}(\Delta J)_{nm}\Big) \\
&\qquad + \frac{1}{4\pi^2\theta^2}\Big(\Delta_{nm;lk}(\Delta J)_{kn}(\Delta J)_{ml} + \Delta_{kn;ml}(\Delta J)_{nm}(\Delta J)_{lk}\Big) \\
&\qquad + \frac{1}{(4\pi^2\theta^2)^2}\Big(\big(\Delta_{nm;kn}\Delta_{lk;ml} + \Delta_{kn;lk}\Delta_{nm;ml}\big) + \Delta_{nm;lk}\Delta_{kn;ml}\Big)\Big\} \\
&+ \frac{\lambda^2}{2(4!)^2} \sum_{m,n,k,l,r,s,t,u\in\mathbb{N}^2} \Big\{\Big[\Big(\Delta_{ml;sr}\Delta_{lk;ts}(\Delta J)_{kn}(\Delta J)_{nm} + \Delta_{ml;sr}\Delta_{kn;ts}(\Delta J)_{lk}(\Delta J)_{nm} \\
&\qquad + \Delta_{ml;sr}\Delta_{nm;ts}(\Delta J)_{lk}(\Delta J)_{kn} + \Delta_{lk;sr}\Delta_{ml;ts}(\Delta J)_{kn}(\Delta J)_{nm} \\
&\qquad + \Delta_{lk;sr}\Delta_{kn;ts}(\Delta J)_{ml}(\Delta J)_{nm} + \Delta_{lk;sr}\Delta_{nm;ts}(\Delta J)_{ml}(\Delta J)_{kn} \\
&\qquad + \Delta_{kn;sr}\Delta_{ml;ts}(\Delta J)_{lk}(\Delta J)_{nm} + \Delta_{kn;sr}\Delta_{lk;ts}(\Delta J)_{ml}(\Delta J)_{nm} \\
&\qquad + \Delta_{kn;sr}\Delta_{nm;ts}(\Delta J)_{ml}(\Delta J)_{lk} + \Delta_{nm;sr}\Delta_{ml;ts}(\Delta J)_{lk}(\Delta J)_{kn} \\
&\qquad + \Delta_{nm;sr}\Delta_{lk;ts}(\Delta J)_{ml}(\Delta J)_{kn} + \Delta_{nm;sr}\Delta_{kn;ts}(\Delta J)_{ml}(\Delta J)_{lk}\Big) \\
&\qquad\qquad \times (\Delta J)_{ru}(\Delta J)_{ut} \\
&\qquad + 5 \text{ permutations of } {}_{ts},\ {}_{sr},\ {}_{ru},\ {}_{ut}\Big] \\
&\qquad + \text{ 1PI-contributions with} \leq 2\ J\text{'s } + \text{ 1PR-contributions }\Big\} + \mathcal{O}(\lambda^3)\;. \qquad (26)
\end{aligned}
$$

In second order in λ we get a huge number of terms so that we display only the 1PI contribution with four J's.

For the classical field (16) we get $\phi^{c\ell}_{mn} = \sum_{p,q\in\mathbb{N}^2} \Delta_{nm;pq}J_{pq} + \mathcal{O}(\lambda)$ so that

$$
J_{pq} = \sum_{r,s\in\mathbb{N}^2} G_{qp;rs}\phi^{c\ell}_{rs} + \mathcal{O}(\lambda)\;. \qquad (27)
$$

The remaining part not displayed in (27) removes the 1PR-contributions when passing to

$\Gamma[\phi^{c\ell}]$. We thus obtain

$$
\begin{aligned}
\Gamma[\phi^{c\ell}] &= \Gamma[0] \\
&\quad + 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2}\Big\{G_{mn;kl} + \frac{\lambda}{6(4\pi^2\theta^2)}\Big(\delta_{ml}\sum_{p\in\mathbb{N}^2}\Delta_{pn;kp} + \delta_{kn}\sum_{p\in\mathbb{N}^2}\Delta_{mp;pl}\Big) && \text{(28a)} \\
&\qquad\qquad + \frac{\lambda}{6(4\pi^2\theta^2)}\Delta_{ml;kn} + \mathcal{O}(\lambda^2)\Big\}\phi^{c\ell}_{mn}\phi^{c\ell}_{kl} && \text{(28b)} \\
&\quad + 4\pi^2\theta^2 \sum_{m,n,k,l,r,s,t,u\in\mathbb{N}^2} \frac{\lambda}{4!}\Big\{\delta_{nk}\delta_{lr}\delta_{st}\delta_{um} && \text{(28c)} \\
&\quad - \frac{\lambda}{2(4!)(4\pi^2\theta^2)}\Big(\sum_{p,q\in\mathbb{N}^2}\big(4\Delta_{mp;qs}\Delta_{pl;tq}\delta_{kn}\delta_{ur} + 4\Delta_{kp;qs}\Delta_{pn;tq}\delta_{ml}\delta_{ur} \\
&\qquad\qquad + 4\Delta_{pl;rq}\Delta_{mp;qu}\delta_{nk}\delta_{st} + 4\Delta_{pn;rq}\Delta_{kp;qu}\delta_{ml}\delta_{st}\big) && \text{(28d)} \\
&\qquad + \sum_{p\in\mathbb{N}^2}\big(4\Delta_{ml;ps}\Delta_{kn;tp}\delta_{ur} + 4\Delta_{kn;ps}\Delta_{ml;tp}\delta_{ur} + 4\Delta_{mp;ts}\Delta_{pl;ru}\delta_{nk} \\
&\qquad\qquad + 4\Delta_{pl;ts}\Delta_{mp;ru}\delta_{nk} + 4\Delta_{kp;ts}\Delta_{pn;ru}\delta_{ml} + 4\Delta_{pn;ts}\Delta_{kp;ru}\delta_{ml} \\
&\qquad\qquad + 4\Delta_{ml;rp}\Delta_{kn;pu}\delta_{st} + 4\Delta_{kn;rp}\Delta_{ml;pu}\delta_{st}\big) && \text{(28e)} \\
&\qquad + \sum_{p,q\in\mathbb{N}^2}\big(4\Delta_{pl;qs}\Delta_{mp;tq}\delta_{nk}\delta_{ur} + 4\Delta_{pn;qs}\Delta_{kp;tq}\delta_{ml}\delta_{ur} \\
&\qquad\qquad + 4\Delta_{kp;rq}\Delta_{pn;qu}\delta_{ml}\delta_{st} + 4\Delta_{mp;rq}\Delta_{pl;qu}\delta_{nk}\delta_{st}\big) && \text{(28f)} \\
&\qquad + 4\Delta_{ml;ts}\Delta_{kn;ru} + 4\Delta_{kn;ts}\Delta_{ml;ru}\Big) + \mathcal{O}(\lambda^2)\Big\}\phi^{c\ell}_{mn}\phi^{c\ell}_{kl}\phi^{c\ell}_{rs}\phi^{c\ell}_{tu} && \text{(28g)} \\
&\quad + \mathcal{O}(\lambda^2)\;.
\end{aligned}
$$

Here, (28a) contains the contribution to the planar two-point function and (28b) the contribution to the non-planar two-point function. Next, (28c) and (28d) contribute to the planar four-point function, whereas (28e), (28f) and (28g) constitute three different types of non-planar four-point functions.

Introducing the cut-off $p^i, q^i \leq \mathcal{N}$ in the internal sums over $p, q \in \mathbb{N}^2$, we split the effective action according to [2] as follows into a relevant/marginal and an irrelevant piece ($\Gamma[0]$ can be ignored):

$$
\Gamma[\phi^{c\ell}] \equiv \Gamma_{\mathrm{rel/marg}}[\phi^{c\ell}] + \Gamma_{\mathrm{irrel}}[\phi^{c\ell}]\;, \tag{29}
$$

$$
\begin{aligned}
\Gamma_{\mathrm{rel/marg}}[\phi^{c\ell}] &= 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2}\Big\{G_{mn;kl} + \frac{\lambda}{6(4\pi^2\theta^2)}\delta_{ml}\delta_{kn}\Big(2\sum_{p^1,p^2=0}^{N}\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}} \\
&\qquad + (m^1{+}n^1{+}m^2{+}n^2)\sum_{p^1,p^2=0}^{N}\big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)\Big) + \mathcal{O}(\lambda^2)\Big\}\phi^{c\ell}_{mn}\phi^{c\ell}_{kl} \\
&\quad + 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{\lambda}{4!}\Big\{1 - \frac{\lambda}{3(4\pi^2\theta^2)}\sum_{p^1,p^2=0}^{N}\big(\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)^2 + \mathcal{O}(\lambda^2)\Big\}\phi^{c\ell}_{mn}\phi^{c\ell}_{nk}\phi^{c\ell}_{kl}\phi^{c\ell}_{lm}\;.
\end{aligned} \tag{30}
$$

To the marginal four-point function and the relevant two-point function there contribute only the projections to planar graphs with vanishing external indices. The marginal two-point function is given by the next-to-leading term in the discrete Taylor expansion around vanishing external indices.

In a regime where $\lambda[\mathcal{N}]$ is so small that the perturbative expansion is valid in (30), the irrelevant part Γ_{irrel} can be completely ignored. Comparing (30) with the initial action according to (2),(7) and (8), we have $\Gamma_{\text{rel/marg}}[\mathcal{Z}\phi^{c\ell}] = S\big[\phi^{c\ell}; \mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}\big]$ with

$$\mathcal{Z} = 1 - \frac{\lambda}{192\pi^2\theta} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2)\;, \tag{31}$$

$$\begin{aligned}\mu^2_{\text{phys}} = \mu_0^2\Big(1 &+ \frac{\lambda}{12\pi^2\theta^2\mu_0^2} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(2\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}} - \Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}}\big) \\ &- \frac{\lambda}{96\pi^2\theta} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2)\Big)\;,\end{aligned} \tag{32}$$

$$\begin{aligned}\lambda_{\text{phys}} = \lambda\Big(1 &- \frac{\lambda}{12\pi^2\theta^2} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)^2 \\ &- \frac{\lambda}{48\pi^2\theta} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2)\Big)\;,\end{aligned} \tag{33}$$

$$\Omega_{\text{phys}} = \Omega\Big(1 + \frac{\lambda(1-\Omega^2)}{192\pi^2\theta\Omega^2} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2)\Big)\;. \tag{34}$$

Solving (32), (33) and (34) for the bare quantities, we obtain to one-loop order

$$\begin{aligned}&\mu_0^2[\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}, \mathcal{N}] \\ &\quad= \mu^2_{\text{phys}}\Big(1 - \frac{\lambda_{\text{phys}}}{12\pi^2\theta^2\mu^2_{\text{phys}}} \sum_{p^1,p^2=0}^{\mathcal{N}} \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}} \\ &\qquad\qquad + \frac{\lambda_{\text{phys}}}{96\pi^2\theta}\Big(1 + \frac{8}{\theta\mu^2_{\text{phys}}}\Big) \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) + \mathcal{O}(\lambda^2_{\text{phys}})\Big)\;,\end{aligned} \tag{35}$$

$$\begin{aligned}&\lambda[\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}, \mathcal{N}] \\ &\quad= \lambda_{\text{phys}}\Big(1 + \frac{\lambda_{\text{phys}}}{12\pi^2\theta^2} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big)^2 + \frac{\lambda_{\text{phys}}}{48\pi^2\theta} \sum_{p^1,p^2=0}^{\mathcal{N}} \big(\Delta_{\substack{1\,p^1\\0\,p^2};\substack{p^1\,1\\p^2\,0}} - \Delta_{\substack{0\,p^1\\0\,p^2};\substack{p^1\,0\\p^2\,0}}\big) \\ &\quad+ \mathcal{O}(\lambda^2_{\text{phys}})\Big)\;,\end{aligned} \tag{36}$$

$$
\begin{aligned}
&\Omega[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\\
&\qquad=\Omega_{\mathrm{phys}}\Big(1-\frac{\lambda_{\mathrm{phys}}(1-\Omega^2_{\mathrm{phys}})}{192\pi^2\theta\Omega^2_{\mathrm{phys}}}\sum_{p^1,p^2=0}^{\mathcal{N}}\big(\Delta_{\substack{1\;p^1\\0\;p^2};\substack{p^1\;1\\p^2\;0}}-\Delta_{\substack{0\;p^1\\0\;p^2};\substack{p^1\;0\\p^2\;0}}\big)+\mathcal{O}(\lambda^2_{\mathrm{phys}})\Big)\;. \qquad (37)
\end{aligned}
$$

Inserting (12) into (36) we can now compute the β_λ-function (23) up to one-loop order, omitting the index $_{\mathrm{phys}}$ on μ^2 and Ω for simplicity:

$$
\begin{aligned}
\beta_\lambda=\frac{\lambda^2_{\mathrm{phys}}}{48\pi^2}\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\sum_{p^1,p^2=0}^{\mathcal{N}}\Bigg\{&\left(\frac{{}_2F_1\Big(\begin{matrix}1\,,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2)\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\end{matrix}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)}{(1+\Omega)^2(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2))}\right)^2\\
&+\frac{p^1(1-\Omega)^2\;{}_2F_1\Big(\begin{matrix}3\,,\;\frac{1+\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2+1)\\ 3+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2+1)\end{matrix}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)}{(1+\Omega)^4\big(\frac{1}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\big)\big(\frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\big)\big(\frac{5}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\big)}\\
&+\frac{{}_2F_1\Big(\begin{matrix}1\,,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2+1)\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2+1)\end{matrix}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)}{2(1+\Omega)^2(\frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2))}-\frac{{}_2F_1\Big(\begin{matrix}1\,,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2)\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\end{matrix}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)}{2(1+\Omega)^2(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2))}+\mathcal{O}(\lambda_{\mathrm{phys}})\Bigg)\Bigg\}\;. \qquad (38)
\end{aligned}
$$

Symmetrising the numerator in the second line $p^1\mapsto\frac{1}{2}(p^1+p^2)$ and using the expansions

$$
\begin{aligned}
{}_2F_1\Big(\begin{matrix}1\,,\;a-p\\ b+p\end{matrix}\Big|z\Big)&=\frac{1}{1+z}+\frac{z(a+b)+z^2(a+b-2)}{p(1+z)^3}+\mathcal{O}(p^{-2})\;,\\
{}_2F_1\Big(\begin{matrix}3\,,\;a-p\\ b+p\end{matrix}\Big|z\Big)&=\frac{1}{(1+z)^3}+\mathcal{O}(p^{-1})\;, \qquad (39)
\end{aligned}
$$

which are valid for large p, we obtain up to irrelevant contributions vanishing in the limit $\mathcal{N}\to\infty$

$$
\begin{aligned}
\beta_\lambda&=\frac{\lambda^2_{\mathrm{phys}}}{48\pi^2}\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\sum_{p^1,p^2=0}^{\mathcal{N}}\frac{1}{(1+\Omega^2_{\mathrm{phys}})^2}\frac{1}{\big(1+p^1+p^2\big)^2}\Big\{1+\frac{(1-\Omega^2_{\mathrm{phys}})^2}{2(1+\Omega^2_{\mathrm{phys}})}-\frac{(1+\Omega^2_{\mathrm{phys}})}{2}\Big\}\\
&\quad+\mathcal{O}(\lambda^3_{\mathrm{phys}})+\mathcal{O}(\mathcal{N}^{-1})\\
&=\frac{\lambda^2_{\mathrm{phys}}}{48\pi^2}\frac{(1-\Omega^2_{\mathrm{phys}})}{(1+\Omega^2_{\mathrm{phys}})^3}+\mathcal{O}(\lambda^3_{\mathrm{phys}})+\mathcal{O}(\mathcal{N}^{-1})\;. \qquad (40)
\end{aligned}
$$

Similarly, one obtains

$$
\beta_\Omega=\frac{\lambda_{\mathrm{phys}}\Omega_{\mathrm{phys}}}{96\pi^2}\frac{(1-\Omega^2_{\mathrm{phys}})}{(1+\Omega^2_{\mathrm{phys}})^3}+\mathcal{O}(\lambda^2_{\mathrm{phys}})+\mathcal{O}(\mathcal{N}^{-1})\;, \qquad (41)
$$

$$
\begin{aligned}
\beta_{\mu_0}&=-\frac{\lambda_{\mathrm{phys}}}{48\pi^2\theta\mu^2_{\mathrm{phys}}(1+\Omega^2_{\mathrm{phys}})}\Big(4\mathcal{N}\ln(2)+\frac{(8+\theta\mu^2_{\mathrm{phys}})\Omega^2_{\mathrm{phys}}}{(1+\Omega^2_{\mathrm{phys}})^2}\Big)\\
&\quad+\mathcal{O}(\lambda^2_{\mathrm{phys}})+\mathcal{O}(\mathcal{N}^{-1})\;, \qquad (42)
\end{aligned}
$$

$$\gamma = \frac{\lambda_{\text{phys}}}{96\pi^2} \frac{\Omega^2_{\text{phys}}}{(1+\Omega^2_{\text{phys}})^3} + \mathcal{O}(\lambda^2_{\text{phys}}) + \mathcal{O}(\mathcal{N}^{-1}) \;. \tag{43}$$

5 Discussion

We have computed the one-loop β- and γ-functions in real four-dimensional duality-covariant noncommutative ϕ^4-theory. Remarkably, this model has a one-loop contribution to the wavefunction renormalisation which compensates partly the contribution from the planar one-loop four-point function to the β_λ-function. The one-loop β_λ-function is non-negative and vanishes in the distinguished case $\Omega = 1$ of the duality-invariant model, see (3). At $\Omega = 1$ also the β_Ω-function vanishes. This is of course expected (to all orders), because for $\Omega = 1$ the propagator (12) is diagonal, $\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\Big|_{\Omega=1} = \frac{\delta_{m^1 l^1}\delta_{k^1 n^1}\delta_{m^2 l^2}\delta_{k^2 n^2}}{\mu_0^2+(4/\theta)(m^1+m^2+n^1+n^2+2)}$, so that the Feynman graphs never generate terms with $|m^i - l^i| = |n^i - k^i| = 1$ in (8).

The similarity of the duality-invariant theory with the exactly solvable models discussed in [4] suggests that also the β_λ-function vanishes to all orders for $\Omega = 1$. The crucial differences between our model with $\Omega = 1$ and [4] is that we are using *real* fields, for which it is not so clear that the construction of [4] can be applied. But the planar graphs of a real and a complex ϕ^4-model are very similar so that we expect identical β_λ-functions (possibly up to a global factor) for the complex and the real model. Since a main feature of [4] was the independence on the dimension of the space, the model with $\Omega = 1$ and matrix cut-off $\mathcal{N}$ should be (more or less) equivalent to a two-dimensional model, which has a mass renormalisation only [8]. Therefore, we conjecture a vanishing β_λ-function in four-dimensional duality-invariant noncommutative ϕ^4-theory to all orders.

The most surprising result is that the one-loop β_Ω-function also vanishes for $\Omega \to 0$. We cannot directly set $\Omega = 0$, because the hypergeometric functions in (38) become singular and the expansions (39) are not valid. Moreover, the power-counting theorems of [2], which we used to project to the relevant/marginal part of the effective action (30), also require $\Omega > 0$. However, in the same way as in the renormalisation of two-dimensional noncommutative ϕ^4-theory [8], it is possible to switch off Ω very weakly with the cut-off $\mathcal{N}$, e.g. with

$$\Omega = \mathrm{e}^{-\big(\ln(1+\ln(1+\mathcal{N}))\big)^2} \;. \tag{44}$$

The decay (44) for large $\mathcal{N}$ over-compensates the growth of any polynomial in $\ln \mathcal{N}$, which according to [2] is the bound for the graphs contributing to a renormalisation of Ω. On the other hand, (44) does not modify the expansions (39). Thus, in the limit $\mathcal{N} \to \infty$, we have constructed the usual noncommutative ϕ^4-theory given by $\Omega = 0$ in (2) at the one-loop level. It would be very interesting to know whether this construction of the noncommutative ϕ^4-theory as the limit of a sequence (44) of duality-covariant ϕ^4-models can be extended to higher loop order.

We also notice that the one-loop β_λ- and β_Ω-functions are independent of the noncommutativity scale θ. There is, however a contribution to the one-loop mass renormalisation via the dimensionless quantity $\mu^2_{\text{phys}}\theta$, see (42).

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